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Note

On the Property of Monotonic Convergence for Multivariate Bernstein-Type Operators*

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We use probabilistic methods to show that a large class of sequences (L_n) of multivariate Bernstein-type operators satisfy the inequality $L_n(f, x) \geqslant L_{n+1}(f, x)$, whenever f is a convex function.

Many sequences (L_n) of one-dimensional approximation operators satisfy the property of monotonic convergence, i.e.,

$$L_n(f, x) \geqslant L_{n+1}(f, x),$$

whenever f is a convex function. This fact is usually shown, for each particular sequence, using methods which depend heavily on the special form of the operators considered (see for instance [3, 6–8, 10, 11]). The technique used by Khan in [7, 8] provides some simple and elegant proofs.

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It is based on a martingale-type property and the conditional version of Jensen's inequality. The aim of this note is to apply such a probabilistic method to the multidimensional case. Moreover, we show that tensor product operators inherit from their "factors" the property of monotonic convergence in a sense to be detailed below.

To begin with, let I be a convex subset of R^k and (L_n) be a sequence of positive linear operators acting on $C(I)$, the space of all real continuous functions defined on I. Suppose that we have the representation

$$L_j(f, x) = Ef(Z_j^x), \qquad x \in I, \quad f \in C(I), \quad j = n, n+1,$$

where E denotes mathematical expectation and Z_n^x, Z_{n+1}^x are I-valued random vectors satisfying the integrability condition

$$E \|Z_j^x\| < \infty, \qquad j = n, n+1,$$

$\|\cdot\|$ being the usual norm in R^k, as well as the martingale-type property

$$E(Z_n^x \mid Z_{n+1}^x) = Z_{n+1}^x \qquad \text{a.s.}, \quad x \in I, \tag{1}$$

where $E(\cdot \mid \cdot)$ denotes conditional expectation [9]. If f is a convex function then we have, by the conditional version of Jensen's inequality [5],

$$\begin{aligned} L_n(f, x) &= E(E(f(Z_n^x) \mid Z_{n+1}^x)) \\ &\geqslant Ef(E(Z_n^x \mid Z_{n+1}^x)) \\ &= Ef(Z_{n+1}^x) \\ &= L_{n+1}(f, x). \end{aligned}$$

In particular, (1) holds when $Z_j^x := (1/j) \sum_{i=1}^{j} U_i^x$, where $U_1^x, ..., U_{n+1}^x$ are integrable, independent identically distributed random vectors with values in I.

We give a few examples to illustrate the application of this method to multivariate Bernstein-type operators. In order to avoid a cumbersome notation we consider the two-dimensional case. The extension to higher dimensions is straightforward.

Examples. (a) Bernstein polynomials on the triangle. Set $I := \{(x, y): x \geqslant 0, y \geqslant 0, x + y \leqslant 1\}$ and let f be a real function on I. The nth Bernstein polynomial of f is defined by

$$\begin{aligned} (B_n f)(x, y) := \sum_{k=0}^{n} \sum_{l=0}^{n-k} f\left(\frac{k}{n}, \frac{l}{n}\right) \frac{n!}{k!\, l!\, (n-k-l)!} \\ \times x^k y^l (1 - x - y)^{n-k-l}. \end{aligned}$$

It is clear that

$$(B_n f)(x, y) = Ef\left(\frac{1}{n}\sum_{i=1}^{n} U_i^{(x,y)}\right),$$

where $U_1^{(x,y)}, U_2^{(x,y)}, \ldots$ are independent random vectors having the same trinomial distribution with parameters $1, x, y$ [4], i.e.,

$$\begin{aligned} P(U_i^{(x,y)} = (k, l)) &= 1 - x - y \qquad && \text{if} \quad (k, l) = (0, 0) \\ &= x && \text{if} \quad (k, l) = (1, 0) \\ &= y && \text{if} \quad (k, l) = (0, 1). \end{aligned}$$

Therefore, if f is convex on I, we have

$$(B_n f)(x, y) \geqslant (B_{n+1} f)(x, y).$$

For an analytic proof of this result see [2].

(b) Baskakov-type operators. Define for $x \geqslant 0$, $y \geqslant 0$, $n = 1, 2, \ldots$

$$\begin{aligned} (B_n^* f)(x, y) := \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} f\left(\frac{k}{n}, \frac{l}{n}\right) \frac{(n+k+l-1)!}{k!\, l!\, (n-1)!} \\ \times x^k y^l (1 + x + y)^{-n-k-l}, \end{aligned}$$

where f is a real function on $[0, \infty) \times [0, \infty)$ such that

$$(B_j^* |f|)(x, y) < \infty, \qquad j = n, n+1.$$

We can write

$$(B_j^* f)(x, y) = Ef\left(\frac{1}{j}\sum_{i=1}^{j} V_i^{(x,y)}\right),$$

where $V_1^{(x,y)}, V_2^{(x,y)}, \ldots$ are independent random vectors having the same negative trinomial distribution with parameters $1, x, y$ [4], i.e.,

$$P(V_i^{(x,y)} = (k, l)) = \binom{k+l}{k} x^k y^l (1 + x + y)^{-1-k-l}, \qquad k, l = 0, 1, 2, \ldots.$$

Therefore, if f is convex, we have

$$(B_n^* f)(x, y) \geqslant (B_{n+1}^* f)(x, y).$$

(c) Operators of Bleimann, Butzer, and Hahn-type. Set $\Delta := \{(x, y): x \geqslant 0,\ y \geqslant 0,\ xy \leqslant 1\}$ and define, for $(x, y) \in \Delta$, $n = 1, 2, \ldots$, and any real function f on Δ

$$(L_n f)(x, y) := \sum_{k=0}^{n} \sum_{l=0}^{n-k} f\left(\frac{k}{n-k+1}, \frac{l}{n-l+1}\right) \frac{n!}{k!\,l!\,(n-k-l)!} \times \left(\frac{x}{1+x}\right)^{k} \left(\frac{y}{1+y}\right)^{l} \left(\frac{1-xy}{(1+x)(1+y)}\right)^{n-k-l}.$$

This is a two-dimensional analogue, distinct from a tensor product, of the operator introduced by Bleimann, Butzer, and Hahn [1]. We have the representation

$$(L_n f)(x, y) = Ef\left(\frac{S_n}{n-S_n+1}, \frac{T_n}{n-T_n+1}\right),$$

where

$$S_n := \sum_{i=1}^{n} U_i, \qquad T_n := \sum_{i=1}^{n} V_i$$

and (U_1, V_1), (U_2, V_2), ... are independent random vectors with the same trinomial distribution with parameters 1, $x/(1+x)$, $y/(1+y)$. It is easy to check that for $k, l = 0, 1, \ldots$, with $k + l \leqslant n+1$

$$\begin{aligned} P(S_n = i, T_n = j \mid S_{n+1} = k, T_{n+1} = l) &= 1 - \frac{k+l}{n+1}, && i = k,\ j = l \\ &= \frac{k}{n+1}, && i = k-1,\ j = l \\ &= \frac{l}{n+1}, && i = k,\ j = l-1. \end{aligned}$$

From this we deduce

$$\begin{aligned} P(S_n = i \mid S_{n+1} = k, T_{n+1} = l) &= \frac{k}{n+1}, && i = k-1 \\ &= 1 - \frac{k}{n+1}, && i = k, \end{aligned}$$

and therefore

$$\begin{aligned} E\left(\frac{S_n}{n-S_n+1} \,\middle|\, S_{n+1} = k, T_{n+1} = l\right) &= \frac{k-1}{n-k+2}\,\frac{k}{n+1} + \frac{k}{n-k+1}\,\frac{n-k+1}{n+1} \\ &= \frac{k}{n+1-k+1}, \end{aligned}$$

if $k \leqslant n$, whereas

$$E\left(\frac{S_n}{n-S_n+1} \,\middle|\, S_{n+1}=n+1,\, T_{n+1}=0\right)=n.$$

Thus, we have

$$E\left(\frac{S_n}{n-S_n+1} \,\middle|\, S_{n+1}, T_{n+1}\right)=\frac{S_{n+1}}{n+1-S_{n+1}+1}-I(S_{n+1}=n+1) \qquad \text{a.s.},$$

where $I(\cdot)$ denotes the indicator function. Similarly

$$E\left(\frac{T_n}{n-T_n+1} \,\middle|\, S_{n+1}, T_{n+1}\right)=\frac{T_{n+1}}{n+1-T_{n+1}+1}-I(T_{n+1}=n+1) \qquad \text{a.s.},$$

and we conclude

$$E(Z_n \mid Z_{n+1})=Z_{n+1}-(I(S_{n+1}=n+1),\, I(T_{n+1}=n+1)) \qquad \text{a.s.},$$

where

$$Z_j:=\left(\frac{S_j}{j-S_j+1}, \frac{T_j}{j-T_j+1}\right).$$

Finally, we can assert the following: Let f be a convex function on $[0, \infty)\times[0, \infty)$ (the convex hull of Δ). If f is nonincreasing in each variable, then

$$(L_n f)(x, y) \geqslant (L_{n+1} f)(x, y), \qquad (x, y) \in \Delta.$$

(d) The examples above are summation operators. As an example of an integral operator we can mention the k-dimensional Weierstrass operator associated with a random vector of the form $x+(1/n)\sum_{i=1}^{n} W_i$, where $W_1, W_2, \ldots$ are independent k-dimensional random vectors having the same multivariate normal distribution with a non-diagonal covariance matrix (not depending on x). The details are omitted.

Finally we show that tensor product operators inherit from their factors the property of monotonic convergence. In order to see this, the integral notation will be more convenient.

Let I (resp. J) be a convex subset of R^k (resp. R^l) and let $(L_m^{(1)})$ (resp. $(L_n^{(2)})$) be a sequence of operators defined by

$$L_m^{(1)}(f, x):=\int_I f\, d\mu_m^x, \qquad x \in I$$

(resp.

$$L_n^{(2)}(g, y):=\int_J g\, dv_n^y, \qquad y \in J),$$

where μ_m^x (resp. ν_n^y) are probability measures on I (resp. J). If $L_{m,n}$ is the tensor product operator defined by

$$(L_{m,n}h)(x, y) := \int_{I\times J} hd(\mu_m^x \otimes \nu_n^y), \qquad (x, y) \in I \times J$$

(where, as usual, "$\otimes$" denotes product measure), then the two following statements are equivalent:

(i) $(L_{m,n}h)(x, y) \geqslant (L_{m+1,n}h)(x, y)$, for any convex function h on $I \times J$ such that $(L_{j,n}|h|)(x, y) < \infty$, $j = m, m+1$.

(ii) $L_m^{(1)}(f, x) \geqslant L_{m+1}^{(1)}(f, x)$, for any convex function f on I such that $L_j^{(1)}(|f|, x) < \infty$, $j = m, m+1$.

Indeed, both implications follow from Fubini's theorem, taking into account the following: If f is a convex function on I, then the function h defined on $I \times J$ by $h(u, v) := f(u)$ is also convex. Conversely, if h is a convex function on $I \times J$, then $h(\cdot, v)$ is a convex function on I, for any $v \in J$.

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640/80/1-11